

On the uniqueness of conical Kähler-Einstein metrics

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Abstract

The purpose of this paper is to prove the uniqueness of conical Kähler-Einstein metrics, under the condition that the twisted *Ding*-functional is proper. This is a generalization of the author's previous work, and we shall first investigate the uniqueness of twisted Kähler-Einstein metrics, and then use these smooth perturbed solutions to approximate the actual conical one. Finally, it will bring the uniqueness of the limiting singular metric.

1 Introduction

The study of Kähler-Einstein equation plays an important role in nowadays geometric analysis after Yau's celebrated work on Calabi's Conjecture[12]. The existence of these metrics were solved on the manifolds with non-positive first Chern class in Yau's paper in 1978[12]. Then Matsushima[10], Futaki[9] found some obstructions for the existence of Kähler-Einstein metrics on Fano manifolds($c_1(X) > 0$), and later Tian[11] gave the first sufficient condition for the existence in the Fano case. Many mathematicians contributed to this question in these 20 years, and most recently Chen, Donaldson and Sun's work[6] proved that the existence of Kähler-Einstein metrics on Fano manifolds is equivalent to the K-stability condition. Their work based on an investigation on some conical Kähler metrics in a special Hölder space $\mathcal{C}^{2,\gamma,\beta}$ for $\gamma < \frac{1}{\beta} - 1$, which was introduced by Donaldson[7].

On the other hand, the uniqueness of Kähler-Einstein metrics on manifolds with $c_1(X) \leq 0$ was proved by Calabi in 1950s, and then Bando-Mabuchi[1] proved the uniqueness of Kähler-Einstein metrics on Fano manifolds($c_1(X) > 0$) in 1987. It's natural to ask questions about uniqueness of conical or even more singular Kähler-Einstein metrics on Fano manifolds. For instance, BBGZ09([2]) proved the uniqueness of Kähler-Einstein metrics starting from a singular setting on a manifold X without any non-trivial holomorphic vector fields. And Berman[3] proved the uniqueness of Donaldson's equation(a generalization of Kähler-Einstein equation in conical setting) based on a technical in Berndtsson's previous work[4] about curvature of log Bergman kernels. And then Berndtsson[5] proved some uniqueness results for Kähler-Einstein metrics with only L^∞ potentials.

The goal of this paper is to prove the uniqueness of $\mathcal{C}^{2,\gamma,\beta}$ conical Kähler-Einstein metrics in another way, based on a new technique developed in the author's previous work[8]. The main ingredients of this technique consist of Chen's $\mathcal{C}^{1,\bar{1}}$ geodesics and a generalization of Futaki's formula. In the first few sections, this technique will be extended to prove the uniqueness of the so-called twisted Kähler-Einstein metrics, i.e. a smooth Kähler metric ω satisfies

$$Ric(\omega) = \omega + \theta$$

where θ is some non-negative closed (1,1) form on X . In fact, assuming the correct cohomology condition(see section 2), we have

Theorem 1 *Suppose ω_0 and ω_1 are two solutions of twisted Kähler-Einstein equation with the same weight θ , then there exists a holomorphic automorphism F on X , such that $F^*(\omega_1) = \omega_0$ and $F^*(\theta) = \theta$, and this automorphism is induced from a holomorphic vector field \mathcal{V} . Moreover, if there is a point $p \in X$ such that the twister θ is strictly positive, then ω_0 is actually fixed, i.e.*

$$\omega_1 = \omega_0$$

on X .

Next, before considering the singular metrics, we shall investigate the perturbed conical Kähler-Einstein equation first, i.e. we put

$$\theta = (1 - \beta)dd^c\chi_\epsilon$$

where the twister $\chi_\epsilon = \log(|s|^2 + \epsilon e^\psi)$ and ψ is some smooth positively curved metric on the line bundle associated with the divisor D . Then we have the uniqueness of the solution of the following equations

$$Ric(\omega_\epsilon) = \omega_\epsilon + dd^c\chi_\epsilon.$$

If we can take the limit when $\epsilon \rightarrow 0$, then in principle, it will bring us the uniqueness of the conical Kähler-Einstein metrics ω_β , i.e. ω_β satisfies

$$Ric(\omega_\beta) = \omega_\beta + (1 - \beta)\delta_D$$

where δ_D is the integration current of the divisor D . However, this is not true in general. In some situations, we can't not even find solutions of the perturbed Kähler-Einstein equations even if the conical Kähler-Einstein metric exists. Hence we need another condition to guarantee the existence of twisted Kähler-Einstein metrics, i.e. the properness of twisted *Ding* functional.

Theorem 2 *Suppose the twisted Ding-functional \mathcal{D}_β is proper, then there is only one $\mathcal{C}^{2,\alpha,\beta}$ solution ω_{φ_β} for the conical Kähler-Einstein equation with angle β along the divisor D on X .*

One direct consequence of above theorem is the uniqueness of Donaldson's equation, i.e.

$$Ric(\omega_\beta) = \beta\omega_\beta + (1 - \beta)\delta_D.$$

for $0 < \beta < 1$. Here the Kähler class is proportional to the anti-canonical class, and the twisted *Ding* functional is automatically proper in this case[7].

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2 Twisted Kähler-Einstein metrics

Let X be a compact complex Kähler manifold with $-K_X > 0$, and S be a semi-positive \mathbb{R} -line bundle on X , i.e. there exists a smooth Hermitian metric ψ on the line bundle S such that

$$\theta = i\partial\bar{\partial}\psi \geq 0.$$

Notice here θ is a globally defined closed $(1, 1)$ form, and then we are going to consider the following twisted Kähler-Einstein equation

$$Ric(\omega) = \omega + \theta \tag{1}$$

on X . Here we assume $-(K_X + S) > 0$, and a Kähler form ω can always be written as

$$\omega = i\partial\bar{\partial}\phi$$

where ϕ is a positively curved smooth metric on the \mathbb{R} -line bundle $-(K_X + S)$. And we will also use another notation when there is no confusion, i.e.

$$\omega_\varphi = \omega_0 + dd^c\varphi$$

where ω_0 is a fixed background Kähler metric in the same cohomology class, and φ is the Kähler potential. We shall consider all such metric with L^∞ potentials, i.e.

$$|\phi' - \phi| < +\infty$$

Let's denote these metrics as $PSH_\infty(-K_X - S)$, and we always assume $\int_X \omega^n = 1$ in the following. Now suppose there exists two Kähler metrics ϕ_0 and ϕ_1 satisfying equation (1), i.e.

$$\omega_{\phi_i}^n = \frac{e^{-\phi_i - \psi}}{\int_X e^{-\phi_i - \psi}} \tag{2}$$

for $i = 1, 2$. We then claim they are the same up to a holomorphic automorphism, i.e.

Theorem 3 Suppose $\omega_1 = i\partial\bar{\partial}\phi_1$ and $\omega_2 = i\partial\bar{\partial}\phi_2$ are two solutions of twisted Kähler-Einstein equation with the same weight θ , then there exists a holomorphic automorphism F on X , such that $F^*(\omega_2) = \omega_1$ and $F^*(\theta) = \theta$. Moreover, this automorphism is induced from a holomorphic vector field \mathcal{V} .

This is the twisted version of uniqueness theorem on Kähler-Einstein metrics on Fano manifolds. In order to investigate this equation from variational methods, we shall introduce the twisted *Ding*-functional, whose critical point corresponds to the twisted Kähler-Einstein metric.

Definition 1 The twisted *Ding*-functional \mathcal{D} is a functional defined on the space of all plurisubharmonic metrics $\phi \in PSH_\infty(-K_X - S)$, such that

$$\mathcal{D} := -\mathcal{E} + \mathcal{F}_\psi$$

where

$$\mathcal{E}(\phi) := \frac{1}{n+1} \int_X \sum_{j=0}^n (\phi - \phi_0) \omega_\phi^j \wedge \omega_{\phi_0}^{n-j}$$

and

$$\mathcal{F}_\psi(\phi) := -\log \int_X e^{-\phi-\psi}.$$

Remark 1 Since ϕ is a metric on the line bundle $-(K_X + S)$ and ψ is a metric on S , we see $\tau = \phi + \psi$ is a metric on $-K_X$, and $e^{-\tau}$ is a volume form on X .

Notice that the two functionals \mathcal{E} and \mathcal{F} will be changed in a new normalization, but *Ding*-functional is a normalization invariant. Now suppose there is a smooth curve ϕ_t in the space of metrics, and we can compute derivatives of twisted *Ding*-functional on it, i.e.

$$\frac{\partial \mathcal{D}}{\partial t} = \int_X \phi' (\omega_\phi^n - \frac{e^{-\phi-\psi}}{\int_X e^{-\phi-\psi}}).$$

Hence twisted Kähler-Einstein metric is its critical point, and

$$\frac{\partial^2 \mathcal{D}}{\partial t^2} = \int_X (\phi'' - |\partial\phi'|_{g_\phi}^2) \omega_\phi^n + (\int_X e^{-\tau})^{-1} \{ \int_X (\phi'' - (\pi_\perp^\tau \phi')^2) e^{-\tau} \}$$

where $\tau = \phi + \psi$, and the orthogonal projection is $\pi_\perp^\tau u = u - \int_X u e^{-\tau} / \int_X e^{-\tau}$. Now suppose $\phi_t, 0 \leq t \leq 1$ is a smooth geodesic connecting ϕ_1 and ϕ_2 , we see

$$(\int_X e^{-\tau}) \frac{\partial^2 \mathcal{D}}{\partial t^2} = \int_X (|\bar{\partial}\phi'|_{g_\tau}^2 - (\pi_\perp^\tau \phi')^2) e^{-\tau} + \int_X (|\bar{\partial}\phi'|_{g_\phi}^2 - |\bar{\partial}\phi'|_{g_\tau}^2) e^{-\tau} \quad (3)$$

where g_τ is the metric corresponding to the Kähler form $\omega_\tau = i\partial\bar{\partial}\tau$, and hence

$$\omega_\tau \geq \omega_\phi.$$

This implies the second term on the RHS of equation(3) is non-negative, i.e. $g_{\phi}^{\alpha\bar{\beta}}\phi'_{,\bar{\beta}}\phi'_{,\alpha} \geq g_{\tau}^{\alpha\bar{\beta}}\phi'_{,\bar{\beta}}\phi'_{,\alpha}$, and the first term is non-negative from the *Futaki's formula* with respect to the weighted Laplacian operator [8]

$$\square_{\tau} = \bar{\partial}_{\tau}^* \bar{\partial}.$$

Hence we proved the following

Proposition 4 *The twisted Ding-functional is convex along smooth geodesics.*

We can further observe that on the smooth geodesic ϕ_t , the twisted *Ding-functional* must keep to be a constant, i.e. $\partial^2 \mathcal{D} / \partial t^2 \equiv 0$ on the geodesic, and then the following two equations

$$\delta_{\tau}(t) := \int_X (|\bar{\partial}\phi'|_{g_{\tau}}^2 - (\pi_{\perp}^{\tau}\phi')^2) e^{-\tau} = 0 \quad (4)$$

$$k(t) := \int_X (|\bar{\partial}\phi'|_{g_{\phi}}^2 - |\bar{\partial}\phi'|_{g_{\tau}}^2) e^{-\tau} = 0 \quad (5)$$

hold simultaneously for each $0 \leq t \leq 1$. The equation (4) implies there exists a time independent holomorphic vector field $V = X^{\alpha} \frac{\partial}{\partial z^{\alpha}}$, where

$$X^{\alpha} = g_{\tau}^{\alpha\bar{\beta}} \phi'_{,\bar{\beta}}$$

such that the Lie derivative

$$\mathcal{L}_{\mathcal{V}}(\tau_t) = 0$$

where $\mathcal{V}_t = V - \frac{\partial}{\partial t}$, i.e. the induced automorphism F preserves the metric as

$$F^*(\omega_{\tau}(t)) = \omega_{\tau}(0)$$

for any $0 \leq t \leq 1$. Then from the twisted Kähler-Einstein equation we see

$$F^*(Ric(\omega_{\phi_1})) = F^*(\omega_{\phi_1} + \theta) = \omega_{\phi_0} + \theta = Ric(\omega_{\phi_0}),$$

hence $(F^*\omega_{\phi_1})^n = (\omega_{\phi_0})^n$ because they are in the same cohomology. By the uniqueness of Monge-Ampère equation, we get

$$F^*(\omega_{\phi_1}) = \omega_{\phi_0},$$

the equation

$$F^*(\theta) = \theta$$

follows directly. Up to here, we proved theorem 3 under the assumption of smooth geodesics. Moreover, the equation (5) implies the twisted Kähler-Einstein metric is really unique if θ is strictly positive. Next, we shall prove the theorem in the case when there is only $\mathcal{C}^{1,\bar{1}}$ geodesic connecting two twisted Kähler-Einstein metrics.

Proof 1 (of theorem 3) *Let's consider the ϵ -approximation geodesics connecting ϕ_1 and ϕ_2 , i.e. the solution of the following equation*

$$(\phi'' - |\partial\phi'|_{g_\phi}^2) \det g_\phi = \epsilon \det h$$

with boundary values

$$\phi(0, z) = \phi_1(z); \quad \phi(1, z) = \phi_2(z).$$

Now if we define $f = \phi'' - |\partial\phi'|_{g_\phi}^2 > 0$, then we can see

$$f \det g_\phi = \epsilon \det h$$

and the second time derivative of twisted Ding-functional is

$$\frac{\partial^2 \mathcal{D}}{\partial t^2} = - \int_X f \det g_\phi + \left(\int_X e^{-\tau} \right)^{-1} \left\{ \int_X (f + \delta_\tau + k) e^{-\tau} \right\}.$$

Notice that $\int_X f_\tau e^{-\tau}$, $\int_X \delta_\tau e^{-\tau}$ and $\int_X k e^{-\tau}$ are all non-negative, hence

$$\int_{X \times I} (f + \delta_\tau + k) e^{-\tau} dt \leq \epsilon C$$

for some uniform constant C independent of ϵ . If put $f_\tau = \phi'' - |\partial\phi'|_{g_\tau}^2 = f + k > 0$, we see that for almost everywhere $t \in [0, 1]$, there exists a constant $C(t)$ and a subsequence $\epsilon_j(t)$ such that

$$\int_X f_\tau e^{-\tau}(\epsilon_j) < C(t)\epsilon_j; \quad \int_X \delta_\tau e^{-\tau}(\epsilon_j) < C(t)\epsilon_j$$

from Fatou's lemma. Furthermore, all the uniform $\mathcal{C}^{1, \bar{1}}$ estimates hold for metrics $\tau(\epsilon)$ on the approximation geodesics, because

$$\tau_\epsilon = \phi_\epsilon + \psi$$

and ψ is a fixed smooth twister here. Hence by considering the weighted Laplacian operator \square_τ , the same argument in [8] implies there exists a time independent holomorphic vector field $V = X^\alpha \frac{\partial}{\partial z^\alpha}$, such that the induced automorphism F of X will preserve the metric, i.e. $F_t^(\omega_\tau(t)) = \omega_\tau(0)$, then as we argued before in smooth case,*

$$F^*(\omega_{\phi_1}) = \omega_{\phi_0}$$

and

$$F^*(\theta) = \theta$$

for all $t \in [0, 1]$.

In prior, the holomorphic vector field $V = X^\alpha \frac{\partial}{\partial z^\alpha}$ obtained from above proof is computed with respect to the metric τ , hence

$$X^\alpha = g_\tau^{\alpha\bar{\beta}} \phi'_{,\bar{\beta}}.$$

However, we know the condition

$$F_t^*(\omega_{\phi_t}) = \omega_{\phi_0} + \theta - F_t^*\theta$$

for each $t \in [0, 1]$ from the proof. Hence the geodesic g_{ϕ_t} is smooth both in space and time directions, and the twisted *Ding*-functional \mathcal{D} keeps being a constant along the geodesic ϕ_t , i.e.

$$\frac{\partial \mathcal{D}}{\partial t} \equiv 0$$

for all $t \in [0, 1]$. Hence ϕ_t keeps to be the local minimizer of the twisted \mathcal{D} -functional along the whole curve, i.e. it satisfies

$$Ric(\omega_{\phi_t}) = \omega_{\phi_t} + \theta.$$

From the same argument as above we have

$$F_t^*(\omega_{\phi_t}) = \omega_{\phi_0}; \quad F_t^*(\theta) = \theta$$

for each $0 \leq t \leq 1$. Then the curve generated by the one parameter group of automorphisms F_t coincides with the geodesic ω_{ϕ_t} , and this is to say the $\mathcal{C}^{1,\bar{1}}$ geodesic is in fact smooth. Moreover, we can prove

Corollary 5 *Suppose there is one point $p \in X$, such that the closed $(1,1)$ form θ is strictly positive, i.e. $\theta(p) > 0$, then the twisted Kähler-Einstein metric ω_1 is actually unique.*

Proof 2 *Suppose we have two different twisted Kähler-Einstein metrics ω_1 and ω_2 , then we can assume there is a smooth geodesic connecting them by the argument before the corollary. And this curve is the one parameter group of Automorphism F_t generated by a nontrivial holomorphic vector field V . Now take the time derivative in the integral equation, we see*

$$F_t^*(\mathcal{L}_V(\omega_{\phi_t})) = 0,$$

and hence

$$\partial(V \lrcorner \omega_{\phi_t}) = \omega'_{\phi_t} = \omega'_\tau$$

which implies

$$\partial(V \lrcorner \theta) = 0$$

Now notice the existence of twisted Kähler-Einstein metrics on the manifold X implies the first betti number $b_1 = 0$, i.e. there is no nontrivial harmonic $(0,1)$ form, hence $V \lrcorner \theta = 0$, i.e.

$$X^\alpha \theta_{\alpha\bar{\beta}} = 0$$

and then we can write

$$X^\alpha = g_\phi^{\alpha\bar{\beta}} \phi'_{\bar{\beta}};$$

for any time $0 \leq t \leq 1$. Now in a neighborhood of the point $p \in U$, the $(1,1)$ form θ keeps to be strictly positive, i.e.

$$\theta > \epsilon \omega$$

in U . But the equation $X^\alpha \theta_{\alpha\bar{\beta}} = 0$ implies the holomorphic vector field V is identically zero in an open set U , hence it must be identically zero on X , which is a contradiction.

Finally by combining the results from theorem 3 and corollary 5, the proof of theorem 1 is finished.

3 Smooth perturbation of conical Kähler-Einstein metrics

We shall introduce conical Kähler-Einstein metrics first in this section, and consider the perturbation of these metrics. Let D be a smooth divisor on the manifold X , such that the associated line bundle S_D of this divisor is semi-positive. Suppose the \mathbb{R} -line bundle $-(K_X + (1 - \beta)S_D)$ is strictly positive, where $0 < \beta < 1$ is any real number. Then each L^∞ strictly plurisubharmonic metric ϕ on this \mathbb{R} -line bundle associates with a Kähler form

$$\omega_\phi = i\partial\bar{\partial}\phi > 0$$

as before.

Definition 2 A singular Kähler metric ω is a conical Kähler metric with angle β along D if the following conditions are satisfied:

- (i) ω is a closed positive $(1,1)$ current on X , and is smooth on $X \setminus D$;
- (ii) for every point $p \in D$, there exists a constant C , such that in $D \cap U = \{z_1 = 0\}$, where U is a coordinate neighborhood of p , we have

$$C^{-1}\omega_\beta \leq \omega \leq C\omega_\beta$$

where ω_β is a local model conical metric on $D \cap U$, i.e.

$$\omega_\beta := \sqrt{-1} \left(\frac{dz^1 \wedge d\bar{z}^1}{|z^1|^{2-2\beta}} + \sum_{i=2}^n dz^i \wedge d\bar{z}^i \right)$$

From now on, we shall suppose ω_ϕ is always a conical Kähler metric, and then we can talk about the Kähler-Einstein equation in this setting: we call ω_ϕ is a *conical Kähler-Einstein metric* if the following equation is satisfied in current sense

$$Ric(\omega_\phi) = \omega_\phi + (1 - \beta)\delta_D \tag{6}$$

where δ_D is the integration current of the divisor D , and it's equivalent to the following Monge-Ampère equation

$$\omega_\phi^n = \frac{e^{-\phi}/|s|^{2-2\beta}}{\int_X e^{-\phi}/|s|^{2-2\beta}}, \quad (7)$$

where $D = \{s = 0\}$. Notice that $e^{-\phi}/|s|^{2-2\beta}$ is a volume form on X by the cohomology condition, since $\log |s|^2$ corresponds to a plurisubharmonic metric on the line bundle S .

Before going to this singular case, it's natural to consider the perturbed version of this equation, i.e. we shall consider the following smooth approximation equation

$$\omega_{\phi_\epsilon}^n = \frac{e^{-\phi_\epsilon}}{\mu_\epsilon(|s|^2 + \epsilon e^\psi)^{1-\beta}} \quad (8)$$

where ψ is any semi-positively curved smooth metric on the line bundle S , and

$$\mu_\epsilon = \int_X \frac{e^{-\phi_\epsilon}}{(|s|^2 + \epsilon e^\psi)^{1-\beta}}$$

is the normalization constant. This is equivalent to the following geometric equality

$$Ric(\omega_{\phi_\epsilon}) = \omega_{\phi_\epsilon} + (1 - \beta)\chi_\epsilon \quad (9)$$

where

$$\chi_\epsilon = dd^c \log(|s|^2 + \epsilon e^\psi)$$

Now we claim that χ_ϵ is an non-negative closed (1,1) form as

Lemma 6 *For any $\epsilon > 0$, we have*

$$dd^c \log(|s|^2 + \epsilon e^\psi) \geq 0.$$

Proof 3

$$\begin{aligned} & \partial\bar{\partial} \log(|s|^2 + \epsilon e^\psi) \\ &= \partial \left\{ \frac{s d\bar{s} + \epsilon e^\psi \bar{\partial}\psi}{|s|^2 + \epsilon e^\psi} \right\} \\ &= \frac{\epsilon e^\psi}{|s|^2 + \epsilon e^\psi} \{ \partial\bar{\partial}\psi + |ds - s\partial\psi|^2 \}, \end{aligned}$$

and notice the above term is non-negative if the smooth metric ψ is.

Now suppose we have a smooth solution ω_{ϕ_ϵ} of equation(8), then theorem(1) tells us

Corollary 7 *The smooth solution of the perturbed conical Kähler-Einstein equation ω_{ϕ_ϵ} is actually unique.*

Proof 4 *It's enough to prove there exist one point $p \in X$, such that $\chi_\epsilon(p) > 0$. First notice that this is the case if the metric $i\partial\bar{\partial}\psi$ is not completely degenerate. Otherwise, we have*

$$\partial\bar{\partial}\psi \equiv 0$$

on X . But meanwhile we should have

$$\bar{\partial}\psi = \frac{d\bar{s}}{\bar{s}}$$

if χ_ϵ vanishes. And then $i\partial\bar{\partial}\psi = \delta_D$, where δ_D is the integration current of D , which is a contradiction.

4 Construction of the perturbed solution

In this section, we shall discuss the existence of the solution of the perturbed Kähler-Einstein equation. In general, this is unknown even if the solution of equation (7) exists. So, here we need an extra assumption: the twisted *Ding*-functional is proper.

In the following, we shall write the twisted *Ding*-functional as \mathcal{D}_ϵ with respect to the smooth twister χ_ϵ , and \mathcal{D}_β with respect to the singular twister $(1 - \beta)\delta_D$, i.e. we have

$$\mathcal{D}_\epsilon(\phi) = -\mathcal{E}(\phi) - \log \int_X \frac{e^{-\phi}}{(|s|^2 + \epsilon e^\psi)^{1-\beta}}$$

and

$$\mathcal{D}_\beta(\phi) = -\mathcal{E}(\phi) - \log \int_X \frac{e^{-\phi}}{|s|^{2-2\beta}}.$$

Notice that the critical point of the twisted *Ding*-functional \mathcal{D}_ϵ (or \mathcal{D}_β) is the solution of the twisted Kähler-Einstein equation with twister χ_ϵ (or $(1 - \beta)\delta_D$). Next, we shall introduce another important functional *Aubin's J* functional, i.e.

$$\mathcal{J}(\phi) := \int_X \varphi \omega_0^n - \mathcal{E}(\varphi)$$

where $\varphi = \phi - \phi_0$. This functional is a kind of $W^{1,2}$ norm of the potential φ , and we can compare it with *Ding*-functional

Definition 3 *The twisted *Ding*-functional \mathcal{D}_ϵ (or \mathcal{D}_β) is called proper if there exists some constant $a > 0$ and b such that*

$$\mathcal{D}_\epsilon(\phi) \geq a\mathcal{J}(\phi) + b$$

for all $\phi \in PSH(-K_X - S)$.

Remark 2 Notice that the properness of the twisted Ding-functional is in fact independent of the twister as long as the twister is smooth. This is because the twisted Ding-functionals are comparable for two different smooth twisters, i.e. let ψ_1 and ψ_2 be the corresponding metrics associated to the twisters θ_1 and θ_2 , then there exists a constant C such that

$$-C < \psi_1 - \psi_2 < C$$

then the twisted functionals satisfy

$$\mathcal{F}_{\psi_2}(\phi) = -\log \int_X e^{-\phi - \psi_2} = -\log \int_X e^{-\phi - \psi_1 + (\psi_1 - \psi_2)}$$

hence

$$|\mathcal{F}_{\psi_1} - \mathcal{F}_{\psi_2}| < C'$$

for some uniform constant C' . And since \mathcal{E} and \mathcal{J} functionals are independent of the twister, the assertion follows.

Remark 3 For the special twister χ_ϵ , observe that the major term \mathcal{F} in the twisted Ding-functional has the following relation

$$\mathcal{F}_\epsilon \leq \mathcal{F}_{\epsilon'}$$

if $\epsilon \leq \epsilon'$, hence \mathcal{D}_ϵ is decreasing when ϵ becomes smaller. Moreover, if $\epsilon = 0$, it achieves its minimum, i.e.

$$\mathcal{D}_\beta \leq \mathcal{D}_\epsilon$$

for all $\epsilon > 0$ small. Hence in the practice, we can simply require \mathcal{D}_β to be proper.

Generally speaking, the purpose for introducing properness of Ding-functional is to solve the following continuity path

$$Ric(\omega_{\phi_t}) = t\omega_{\phi_t} + (1-t)\omega_0 + (1-\beta)\chi_\epsilon.$$

where ω_0 is a fixed smooth Kähler metric in the same cohomology with ω_ϕ , and $\chi_\epsilon = i\partial\bar{\partial}\psi_\epsilon$ for $\psi_\epsilon = \log(|s|^2 + \epsilon e^\psi)$. This equation is solvable up to $t = 1$ if the twisted K -energy [3] is proper, and this condition is equivalent to the properness of the twisted Ding-functional from the argument of Berman[3], so the existence of smooth perturbed solutions are guaranteed from here. However, this is not the end of the story since we want to make sure these perturbed solutions can approximate the conical one in some sense.

In order to do this, we need to consider a kind of conical Kähler metrics with better regularity. Donaldson[7] introduced a special Hölder space $\mathcal{C}^{2,\alpha,\beta}$ where $0 < \alpha < \frac{1}{\beta} - 1$ for real valued functions on X , and we can define the so called $\mathcal{C}^{2,\alpha,\beta}$ conical Kähler metric[7] by requiring that a local Kähler potential lies in $\mathcal{C}^{2,\alpha,\beta}$, i.e. in a local coordinate chart near the divisor D , we can always write

$$\omega = i\partial\bar{\partial}(\varphi + \psi)$$

where $\varphi \in \mathcal{C}^{2,\alpha,\beta}$, and ψ is some smooth function. Be aware that this condition is stronger than the condition of conical Kähler metrics. Simply speaking, for fixed angle β , a $\mathcal{C}^{2,\alpha,\beta}$ conical Kähler metric is a conical Kähler metric with uniform $\mathcal{C}^{2,\alpha,\beta}$ norm.

Now let's take ω_{φ_β} to be a $\mathcal{C}^{2,\alpha,\beta}$ conical Kähler-Einstein metrics on X and assume the twisted *Ding*-functional \mathcal{D}_β is proper, then Chen, Donaldson and Sun's work[6] provided a way to construct the following a family of sequence of perturbed solutions

$$\omega_{\phi_\epsilon}(t, \cdot) = \omega_0 + i\partial\bar{\partial}\phi_\epsilon,$$

and it satisfies the following equation

$$\text{Ric}(\omega_{\phi_\epsilon(t)}) = t\omega_{\phi_\epsilon(t)} + (1-t)\omega_{\varphi_\epsilon} + (1-\beta)\chi_\epsilon.$$

Notice that for $t = 1$, $\omega_{\phi_\epsilon}(1, \cdot) = \omega_{\phi_\epsilon}$ is exactly the solution of the twisted Kähler-Einstein equation with smooth twister $(1-\beta)\chi_\epsilon$, i.e.

$$\text{Ric}(\omega_{\phi_\epsilon}) = \omega_{\phi_\epsilon} + (1-\beta)\chi_\epsilon,$$

and when $t = 0$, the metric $\omega_{\phi_\epsilon}(0, \cdot) = \psi_\epsilon$ will approximate the original metric ω_β because we require

$$\omega_{\psi_\epsilon}^n = e^{-\varphi_\epsilon + h\omega_0} \frac{1}{(|s|_h^2 + \epsilon)^{1-\beta}} \omega_0^n$$

where $\omega_{\varphi_\epsilon}$ is a small perturbation of the original metric such that $\omega_{\varphi_\epsilon} \rightarrow \omega_\beta$ in $C^\gamma(X)$. Moreover, we know that when $\epsilon \rightarrow 0$, the sequence of metrics $\phi_\epsilon(t, \cdot)$ converges to a metric $\phi_0(t, \cdot)$ globally in C^γ and locally in $C^{3,\gamma}$ outside the divisor D , and the limiting metric $\phi_0(t, \cdot)$ will satisfy the following equation outside D

$$\omega_{\phi_0}^n = e^{-t\phi_0 - (1-t)\varphi_\beta + h\omega_0} \frac{1}{|s|_h^{2-2\beta}} \omega_0^n \quad (10)$$

with

$$\phi_0(0, \cdot) = \varphi_\beta.$$

Observe that above equation is in fact equivalent to

$$\omega_{\phi_0}^n = e^{-t(\phi_0 - \varphi_\beta)} \omega_{\varphi_\beta}^n \quad (11)$$

with $t \in [0, 1]$. Now we claim the curve of metrics $\phi_0(t, 0)$ is in fact fixed. The argument of the claim is similar with the end of the paper [6], and we shall recall it here for convenience of the reader

Lemma 8 *The continuous family $\phi_0(t, \cdot)$ is independent of the time t , and hence*

$$\phi_0(1, \cdot) = \varphi_\beta.$$

Proof 5 We shall argue like the end of the paper [6]. First notice that the weighted Laplacian operator Δ_{φ_β} is continuous and invertible as a map

$$\Delta_{\varphi_\beta} : \mathcal{C}_0^{2,\gamma,\beta} \rightarrow \mathcal{C}^{\gamma,\beta}$$

for some $\gamma < \frac{1}{\beta} - 1$, and $\mathcal{C}_0^{2,\gamma,\beta}$ is the space of functions in $\mathcal{C}^{2,\gamma,\beta}$ with zero average. In fact, by Donaldson's Hölder estimate of conical metrics in [7], we can prove the first eigenvalue of this operator is strictly positive, i.e.

$$\Delta_{\varphi_\beta} > \lambda$$

for some constant $\lambda > 0$, and any $u \in \mathcal{C}_0^{2,\gamma,\beta}$. Now since the term $e^{-t(\phi_0 - \varphi_\beta)}$ lies in $\mathcal{C}^{\gamma,\beta}$, Implicit Function Theorem implies the existence of a continuous family of solution of the following equation

$$\omega_\psi^n = e^{-t(\phi_0 - \varphi_\beta)}$$

with $\psi(t, \cdot) \in \mathcal{C}_0^{2,\gamma,\beta}$, for $t \in [0, \epsilon_0)$. Notice that $\psi(0, \cdot) = \varphi_\beta$, hence $\psi(t, \cdot)$ must coincide with $\phi_0(t, \cdot)$ up to a constant in this short time. Then we can guarantee that $\phi_0(t, \cdot) \in \mathcal{C}^{2,\gamma,\beta}$ for $t \in [0, \epsilon_0)$.

Next we consider a constant continuity path $\varphi_t \equiv \varphi_\beta$, which satisfies the equation

$$\omega_{\varphi_t}^n = e^{-t(\varphi_t - \varphi_\beta)} \omega_{\varphi_\beta}^n.$$

Now by Implicit Function Theorem again, we have a unique path of solution for $t < \min(\epsilon_0, \lambda/2)$, and hence

$$\phi_0(t, \cdot) = \varphi_\beta, \quad \forall 0 \leq t < \min(\epsilon_0, \lambda/2).$$

Finally we can repeat this procedure again and again, it will reach $t = 1$, i.e.

$$\phi_0(1, \cdot) = \varphi_\beta.$$

Recall from our previous construction, we see the sequence of twisted Kähler-Einstein metric ϕ_ϵ will converges to φ_β in C^γ , hence we have

Proposition 9 Suppose ω_{φ_β} is a $\mathcal{C}^{2,\alpha,\beta}$ solution for the conical Kähler-Einstein equation along the divisor D on X , and the twisted Ding-functional is proper, we can construct a sequence of perturbed Kähler-Einstein metrics ϕ_ϵ , such that

$$\phi_\epsilon \rightarrow \varphi_\beta$$

globally in C^γ , and locally in $C^{k,\gamma}$ for some $k > 3$ outside of D .

and this brings the proof our uniqueness.

Proof 6 (of theorem 2) Suppose ω_{φ_β} and ω'_{φ_β} are two different $\mathcal{C}^{2,\alpha,\beta}$ conical Kähler-Einstein metrics on X with cone angle β along the divisor D . From above argument, we can construct a sequence of smooth twisted Kähler-Einstein metrics ϕ_{ϵ_j} to approximate φ_β in some Hölder space C^γ . Then repeat the construction again, we can find a subsequence $\phi_{\epsilon_{n(j)}}$ to approximate φ'_β by the actual uniqueness of ϕ_ϵ , and this implies $\phi_\beta = \phi'_\beta$ on X , which is a contradiction.

In the end of the paper, a new point in the above proof should be mentioned. Unlike the end of the paper[6], the vanishing of tangential holomorphic vector fields is not necessary in the proof of lemma(8), i.e. the properness of twisted *Ding* functional has not been used here. So we can try to generalize this method by starting from the construction of some suitable approximation of conical Kähler-Einstein metrics without properness of *Ding* functional.

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